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RESEARCH PROBLEMS

The purpose of the research problems section is the presentation of unsolved problems in discrete mathematics. Older problems are acceptable if they are not as widely known as they deserve. Problems should be submitted using the format as they appear in the journal and sent to

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Readers wishing to make comments dealing with technical matters about a problem that has appeared should write to the correspondent for that particular problem. Comments of a general nature about previous problems should be sent to Professor Alspach.

Problem 130. Posed by: Dino J. Lorenzini.

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Let G be a connected graph with m edges, n vertices and adjacency matrix $A=((a_{ij}))$. Let d_i denote the degree of the ith vertex and define the Laplacian of G to be the matrix $M:=D-A$ with $D=\operatorname{diag}(d_1,\ldots,d_n)$. Let ${}^tJ=(1,\ldots,1):Z^n\to Z$. We define $\Phi:=\operatorname{Ker}{}^tJ/Im\,M$. The finite abelian group Φ can be described in terms of the Smith Normal Form $F=\operatorname{diag}(\varphi_1,\ldots,\varphi_{n-1},0)$ of M. Any diagonal matrix $E=\operatorname{diag}(e_1,\ldots,e_{n-1},0)$ row and column equivalent to M over the integers induces an isomorphism $\Phi\approx Z/e_1Z\times\cdots\times Z/e_{n-1}Z$ (see [3]). The integers $\varphi_1|\cdots|\varphi_{n-1}$ can be computed in the following way: $\varphi_i=\Delta_i/\Delta_{i-1}$ where $\Delta_0=1$ and Δ_i is the gcd of the determinants of the $i\times i$ minors of M. The integer $|\Phi|=\Delta_{n-1}=\varphi_1\cdots\varphi_{n-1}$ is well known: it equals the number of spanning trees of G (see [2]).

Fact 1 (See [3, Section 5] or [4]). *Let $\beta(G) = m - (n-1)$ be the number of independent cycles of G. If $\beta \leq n-1$, then $\Delta_{n-1-\beta} = 1$.*

For any graph G, define the integer $\tilde{\beta}$ by the formula $\tilde{\beta} = \beta - \sum_{i<j, a_{ij}\neq 0} (a_{ij} - 1)$. $\tilde{\beta}$ is the first Betti number of the complex obtained from G by spanning a 2-cell into every polygon having two sides. It follows from the proof of 6.1 in [3] that it is possible to bound the product $\varphi_1 \cdots \varphi_{n-1-\tilde{\beta}} = \Delta_{n-1-\tilde{\beta}}$ in an explicit way. We say that two spanning trees T_1 and T_2 of G are equivalent if the following condition holds for all i, j: "v_i is linked to v_j in T_1 iff v_i is linked to v_j in T_2". We asssociate to an equivalence class T of spanning trees the integer $c(T) := \prod a_{ij}$, where the product is taken over all pairs $i<j$ such that v_i is linked to v_j in T.

Fact 2. *If $\tilde{\beta} \leq n-1$, then $\Delta_{n-1-\tilde{\beta}}$ divides the greatest common divisor of the integers $c(T)$, T an equivalence class of spanning trees of G.*

It is interesting to note that the integer Δ_{n-1}, i.e. the number of spanning trees of G, can be computed as $\Delta_{n-1} = \Sigma c(T)$. In [1], top of page 378, Artin and Winters define the integer $\bar{\beta}$ to be the first Betti number of the complex obtained from G by spanning a 2-cell into every polygon having two or three vertices. They show:

Fact 3. *If $\bar{\beta} \leq n-1$, then $\Delta_{n-1-\bar{\beta}}$ is uniformally bounded, for all graphs G, by an integer c depending only β.*

Problem. If β_i is defined to be the first Betti number of the complex obtained from G by spanning a 2-cell into every polygon having at most i vertices, does $\Delta_{n-1-\beta_i}$ divide a constant c depending only on β?

To conclude this note, we present some explicit computations of the group Φ, obtained by means of rows and columns operations on the matrix M; note that the described splittings of Φ are not always the canonical ones.

0. The complete graph on n vertices has $\Phi \cong (\mathbb{Z}_n)^{n-2}$.
1. Let G be a graph obtained from a complete graph on n vertices by removing all edges linking together k vertices ($2 \leq k < n$).
$$\Phi \cong (\mathbb{Z}_n)^{n-k-2} \times (\mathbb{Z}_{(n-k)})^{k-2} \times \mathbb{Z}_{n(n-k)}.$$
2. Let G be the graph obtained from a complete graph on n vertices by removing k edges linking a given vertex ($2 \leq k < n-2$).
$$\Phi \cong (\mathbb{Z}_n)^{n-k-3} \times (\mathbb{Z}_{n-1})^{k-2} \times \mathbb{Z}_{n(n-1)(n-k-1)}.$$
3. Let X and Y be sets of k vertices and l vertices respectively and $n = k + l$ (k, $l \geq 2$). Consider the graph G obtained by linking in all possible ways the

vertices in X to the vertices in Y but allowing no edge between two vertices of X or of Y.

$$\Phi \cong (\mathbb{Z}_k)^{l-2} \times (\mathbb{Z}_l)^{k-2} \times \mathbb{Z}_{kl}.$$

4. Let X and Y be two complete graphs on k vertices ($k \geqslant 2$). Take k extra edges and link the vertices of X to the vertices of Y so that the resulting graph is k-regular.

$$\Phi \cong \mathbb{Z}_{k+2} \times (\mathbb{Z}_{k(k+2)})^{k-2}.$$

References

[1] M. Artin and G. Winters, Degenerate fibers and stable reduction of curves, Topology 10 (1971) 373–383.
[2] N. Biggs, Algebraic Graph Theory (Cambridge University Press, Cambridge, UK, 1974).
[3] D. Lorenzini, Arithmetical graphs, preprint, 1988.
[4] D. Lorenzini, On a finite group attached to the laplacian of a graph, preprint, 1988.

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See the preceding problem for appropriate definitions and notation.

Problem. Can the bound c above be given explicitly? In case of $\bar{\beta}$, $\Delta_{n-1-\bar{\beta}}$ is bounded by $2^{\beta-\bar{\beta}}$. Let D be a simply connected region in the plane and let G be a graph describing a triangulation of D, so that $\bar{\beta}=0$. The number of spanning trees of G is bounded by 3^{β} and this bound is achieved.